Exo 12)

$$r(\vec{x}) = \sqrt{x_1^2 + \dots + x_n^2} = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$$

1. Soit $i \in [\![1, n]\!]$, $\vec{x} \in \mathbb{R} \setminus \{\vec{0}\}$

$$\frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\left(\sum_{j=1}^{n} x_j^2\right)^{1/2}\right]$$

$$= \frac{1}{2} \cdot \frac{2 x_i}{\left(\sum_{j=1}^{n} x_j^2\right)^{1/2}} \qquad \left((\sqrt{u})' = \frac{1}{2}\frac{u'}{\sqrt{u}}\right)$$

Soit : $\boxed{\frac{\partial r}{\partial x_i} = \frac{x_i}{r}}$ (Rq : attention $r(\vec{x})$)

2. Soit $f : \mathbb{R}^n \to \mathbb{R}$ telle que $f(\vec{x}) = \Psi(r)$

Donc $f(\vec{x}) = \Psi \circ r(\vec{x})$ et $\Psi : \mathbb{R}^+ \to \mathbb{R}$.

* $$\frac{\partial f}{\partial x_i} = \frac{\partial \Psi}{\partial r} \cdot \frac{\partial r}{x_i} = \Psi'(r) \cdot \frac{\partial r}{\partial x_i} = \Psi'(r) \frac{x_i}{r}$$

* $$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\Psi'(r) \frac{x_i}{r}\right)$$

$$= \frac{\partial}{\partial x_i}(\Psi'(r)) \cdot \frac{x_i}{r} + \Psi'(r) \cdot \frac{\partial}{\partial x_i}\left(\frac{x_i}{r(\vec{x})}\right)$$

$$= \underbrace{\frac{\partial \Psi'(r)}{\partial r}}_{\Psi''(r)} \cdot \underbrace{\frac{\partial r}{\partial x_i}}_{\frac{x_i}{r}} \cdot \frac{x_i}{r} + \Psi'(r) \cdot \frac{r - x_i \frac{\partial r}{\partial x}}{r^2}$$

$$= \Psi''(r) \left(\frac{x_i}{r}\right)^2 + \Psi'(r) \cdot \frac{r - \frac{x_i^2}{r}}{r^2}$$

<u>Donc</u> : $\displaystyle \underbrace{\frac{\partial^2 f}{\partial x_i^2} = \varphi''(r)\left(\frac{x_i}{r}\right)^2 + \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)\varphi'(r)}$

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} = \frac{\varphi''(r)}{r^2}\cdot \underbrace{\sum_{i=1}^{n} x_i^2}_{=r^2} + \varphi'(r)\sum_{i=1}^{n} \frac{1}{r} - \frac{x_i^2}{r^3}$$

$$= \varphi''(r)\cdot \cancel{\frac{r^2}{r^2}} + \varphi'(r)\left(\frac{1}{r}\cdot \underbrace{\sum_{i=1}^{n} 1}_{=n} - \frac{1}{r^3}\underbrace{\sum_{i=1}^{n} x_i^2}_{=r^2}\right)$$

$$\boxed{\Delta f = \varphi''(r) + \frac{n-1}{r}\varphi'(r)}$$

3. $\Delta f = 0$ sur $\Omega = \mathbb{R}^n \setminus \{\vec{0}\} \iff \varphi''(r) + \frac{n-1}{r}\varphi'(r) = 0$ <u>sur $\mathbb{R}^{+*}$</u>.

① <u>Si $n=1$:</u> $\varphi''(r) = 0$

<u>$\varphi(r) = C_1 r + C_2 \quad , \; C_1, C_2 \in \mathbb{R}$.</u>

② <u>Si $n \geq 2$:</u>

$$\frac{\varphi''(r)}{\varphi'(r)} = \frac{1-n}{r}$$

$$\ln|\varphi'(r)| = (1-n)\ln r + K$$

$$\Psi'(r) = C_1 \; r^{(1-n)} \qquad C_1 \in \mathbb{R} \; (= \pm e^k)$$

ⓐ Si $n = 2$: $\quad \Psi'(r) = \dfrac{C_1}{r}$

$$\Psi(r) = C_1 \ln r + C_2 \quad , \quad C_1, C_2 \in \mathbb{R}$$

ⓑ Si $n > 2$: $\quad \Psi'(r) = C_1 \, r^{1-n}$

$$\Psi(r) = C_1 . \frac{r^{2-n}}{2-n} + C_2 \quad , \quad C_1, C_2 \in \mathbb{R}$$