Ex 4) $\vec{f}(x,y,z) = \begin{pmatrix} xy \\ yz \\ xz \end{pmatrix}$

$\Omega = \left\{ (x,y,z) \in \mathbb{R}^3 : 0 < z < 1-x-y,\ 0<y<1-x,\ 0<x<1 \right\}$

Vérifions : $\underbrace{\iint_{\partial\Omega} \vec{f}.\vec{ds}}_{I_1} = \underbrace{\iiint_{\Omega} \operatorname{div}\vec{f}\, dv}_{I_2}$

① $\operatorname{div}\vec{f} = y + z + x$

$I_2 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x+y+z \; dz\,dy\,dx$

$= \int_0^1 \int_0^{1-x} \left[xz + yz + \frac{z^2}{2} \right]_0^{1-x-y} dy\,dx$

$= \int_0^1 \int_0^{1-x} x(1-x-y) + y(1-x-y) + \frac{(1-x-y)^2}{2} \; dy\,dx$

$= \int_0^1 \int_0^{1-x} \frac{1}{2}\left(2x+2y + 1-x-y\right)(1-x-y)\; dy\,dx$

$= \frac{1}{2}\int_0^1 \int_0^{1-x} (1+x+y)(1-x-y)\; dy\,dx$

$= \frac{1}{2}\int_0^1 \int_0^{1-x} 1-(x+y)^2 \; dy\,dx$

$= \frac{1}{2}\int_0^1 \left[y\right]_0^{1-x} - \left[\frac{(x+y)^3}{3}\right]_0^{1-x} dx$

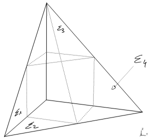
$= \frac{1}{2}\int_0^1 1-x - \frac{1}{3}\left(1-x^3\right) dx$

$= \frac{1}{2}\int_0^1 \frac{2}{3} - x + \frac{1}{3}x^3 \; dx$

$= \frac{1}{2}\left(\frac{2}{3} - \frac{1}{2} + \frac{1}{3}\cdot\frac{1}{4}\right)$

$= \frac{1}{2}\cdot\left(\frac{8-6+1}{12}\right) = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}$

②



$\partial\Omega = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$

$I_1 = \iint_{\partial\Omega} \vec{f}.\vec{ds}$

$= \underbrace{\iint_{\Sigma_1} \vec{f}.\vec{ds}}_{\Phi_1} + \underbrace{\iint_{\Sigma_2} \vec{f}.\vec{ds}}_{\Phi_2} + \underbrace{\iint_{\Sigma_3} \vec{f}.\vec{ds}}_{\Phi_3} + \underbrace{\iint_{\Sigma_4} \vec{f}.\vec{ds}}_{\Phi_4}$

① Σ_1 : $\vec{\sigma_1}(x,z) = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$, $x \in [0,1]$, $z \in [0,1-x]$

$\Phi_1 = \int_0^1 \int_0^{1-x} \underbrace{\vec{f}(\vec{\sigma_1}(x,z))}_{\begin{pmatrix} 0 \\ 0 \\ xz \end{pmatrix}} \cdot \underbrace{\left(\frac{\partial\vec{\sigma_1}}{\partial x} \wedge \frac{\partial\vec{\sigma_1}}{\partial z}\right)}_{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\wedge\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} dz\,dx$

produit scalaire $= 0 \implies \Phi_1 = 0$

② Σ_2 : $\vec{\sigma_2}(x,y) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$, $x \in [0,1]$, $y \in [0,1-x]$

$\Phi_2 = \int_0^1 \int_0^{1-x} \underbrace{\vec{f}(\vec{\sigma_2}(x,y))}_{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} \cdot \underbrace{\frac{\partial\vec{\sigma_2}}{\partial x} \wedge \frac{\partial\vec{\sigma_2}}{\partial y}}_{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\wedge\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} dy\,dx$

produit scalaire $= 0 \implies \Phi_2 = 0$

③ Σ_3 : $\vec{\sigma}_3(y,z) = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$, $y \in [0,1]$, $z \in [0, 1-y]$

$$\Phi_3 = \int_0^1 \int_0^{1-y} \vec{f}(\vec{\sigma}_3(y,z)) \cdot \frac{\partial \vec{\sigma}_3}{\partial y} \wedge \frac{\partial \vec{\sigma}_3}{\partial z} \, dz \, dy$$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

produit scalaire nul $\Rightarrow \Phi_3 = 0$

④ Σ_4 : $\vec{\sigma}_4(x,y) = \begin{pmatrix} x \\ y \\ 1-x-y \end{pmatrix}$, $x \in [0,1]$, $y \in [0, 1-x]$

$$\Phi_4 = \int_0^1 \int_0^{1-x} \vec{f}(\vec{\sigma}_4(x,y)) \cdot \frac{\partial \vec{\sigma}_4}{\partial x} \wedge \frac{\partial \vec{\sigma}_4}{\partial y} \, dy \, dx$$

$\begin{pmatrix} xy \\ y(1-x-y) \\ x(1-x-y) \end{pmatrix}$ $\quad$ $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\Phi_4 = \int_0^1 \int_0^{1-x} \cancel{xy} + y - xy - y^2 + x - x^2 - \cancel{xy} \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} (1-x)y - y^2 + x(1-x) \, dy \, dx$$

$$= \int_0^1 (1-x) \cdot \left[\frac{y^2}{2}\right]_0^{1-x} - \left[\frac{y^3}{3}\right]_0^{1-x} + x(1-x)\left[y\right]_0^{1-x} dx$$

$$= \int_0^1 \frac{1}{2}(1-x)^3 - \frac{(1-x)^3}{3} + x(1-x)^2 \, dx$$

$$= \int_0^1 \frac{1}{6}(1-x)^3 + x(1-x)^2 \, dx$$

$$= \frac{1}{6}\left[\frac{-(1-x)^4}{4}\right]_0^1 + \int_0^1 x - 2x^2 + x^3 \, dx$$

$$= \frac{1}{24}(+1) + \frac{1}{2} - 2\frac{1}{3} + \frac{1}{4}$$

$$= \frac{1}{24} + \frac{12}{24} - \frac{16}{24} + \frac{6}{24} = \frac{3}{24} = \frac{1}{8}$$

Finalement : $I_1 = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4$

$= 0 + 0 + 0 + \frac{1}{8}$

$$\boxed{I_1 = \frac{1}{8} = I_2}$$