TD3 Exo 5

$$\vec{f}(x,y,z) = \begin{pmatrix} xy \\ yz \\ xz \end{pmatrix}$$

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : 0 < z < 1-x-y,\ 0 < y < 1-x,\ 0 < x < 1\}$$

A-t'on bien : $\underbrace{\iiint_{\Omega} \operatorname{div} \vec{f} \, dx\,dy\,dz}_{I_1} \overset{?}{=} \underbrace{\iint_{\Sigma = \partial\Omega} \vec{f} \cdot d\vec{S}}_{I_2}$

⇝ Schéma à la fin

→ $\operatorname{div} \vec{f} = y + z + x$

$$I_1 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y + z + x \, dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (y+x)\left[z\right]_0^{1-x-y} + \left[\frac{z^2}{2}\right]_0^{1-x-y} dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (y+x)(1-x-y) + \frac{1}{2}(1-x-y)^2 \, dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}(1-x-y)(\underbrace{2y + 2x + 1 - x - y}_{1+x+y}) \, dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}\left[1-(x+y)\right]\left[1+(x+y)\right] dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}\left[1-(x+y)^2\right] dy\,dx$$

(2)

$$I_1 = \frac{1}{2} \int_0^1 \left[y\right]_0^{1-x} - \left[\frac{1}{3}(x+y)^3\right]_0^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 1-x - \frac{1}{3}\left(1-x^3\right) dx$$

$$= \frac{1}{2} \int_0^1 \frac{2}{3} - x + \frac{x^3}{3} \, dx$$

$$= \frac{1}{2}\left(\frac{2}{3} - \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{4} \right)$$

$$= \frac{1}{2}\left(\frac{8-6+1}{12} \right) = \frac{1}{2} \cdot \frac{3}{12} = \frac{1}{2} \cdot \frac{1}{4} = \underline{\frac{1}{8}}.$$

$\rightarrow$ Pour $I_2 = \iint_\Sigma \vec{f} \cdot d\vec{S} = \underbrace{\iint_{\Sigma_1} \vec{f} \cdot d\vec{S_1}}_{I_{21}} + \underbrace{\iint_{\Sigma_2} \vec{f} \cdot d\vec{S_2}}_{I_{22}} + \underbrace{\iint_{\Sigma_3} \vec{f} \cdot d\vec{S_3}}_{I_{23}} + \underbrace{\iint_{\Sigma_4} \vec{f} \cdot d\vec{S_4}}_{I_{24}}$

$\rightsquigarrow$ Attention, les $\vec{dS_i}$ sont dirigés selon la normale sortante ...

<u>Paramétrisations:</u>

- Σ_1 : $\vec{\sigma_1}(x,z) = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$, $\quad 0 < x < 1$, $\quad 0 < z < 1-x$

$$\frac{\partial \vec{\sigma_1}}{\partial x} \wedge \frac{\partial \vec{\sigma_1}}{\partial z} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$ $\rightsquigarrow$ normale sortante à Σ_1 $\boxed{\text{OK}}$

$$\vec{dS_1} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} dx\, dz$$

$$I_{21} = \int_0^1 \int_0^{1-x} \vec{f}(\vec{\sigma}(x,z)) \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} dx\, dz = \int_0^1 \int_0^{1-x} \begin{pmatrix} 0 \\ 0 \\ xz \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} dx\, dz = 0$$

⊛ Σ_2 : $\vec{\alpha_2}(x,y) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$, $\quad 0 < x < 1$, $\quad 0 < y < 1-x$

$$\frac{\partial \vec{\alpha_2}}{\partial x} \wedge \frac{\partial \vec{\alpha_2}}{\partial y} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longrightarrow \text{normale entrante}$$

donc : $\vec{dS_2} = - \left(\frac{\partial \vec{\alpha_2}}{\partial x} \wedge \frac{\partial \vec{\alpha_2}}{\partial y}\right) dx\,dy = - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dx\,dy$

$$I_{22} = - \int_0^1 \int_0^{1-x} \underbrace{\vec{f}(\vec{\alpha_2}(x,y))}_{\begin{pmatrix} xy \\ 0 \\ 0 \end{pmatrix}} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dx\,dy = \underline{0}$$

⊛ Σ_3 : $\vec{\alpha_3}(y,z) = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$ $\quad 0 < y < 1$, $\quad 0 < z < 1-y$ $\quad (x=0)$

$$\frac{\partial \vec{\alpha_3}}{\partial y} \wedge \frac{\partial \vec{\alpha_3}}{\partial z} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \text{normale entrante donc:}$$

$$\vec{dS_3} = - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dy\,dz$$

$$I_{23} = - \int_0^1 \int_0^{1-y} \underbrace{\vec{f}(\vec{\alpha_3}(y,z))}_{\begin{pmatrix} 0 \\ yz \\ 0 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dy\,dz = \underline{0}$$

⊛ Σ_4 : $\vec{\alpha_4}(x,y) = \begin{pmatrix} x \\ y \\ 1-(x+y) \end{pmatrix}$ $\quad 0 < x < 1$, $\quad 0 < y < 1-x$.

$$\frac{\partial \vec{\alpha_4}}{\partial x} \wedge \frac{\partial \vec{\alpha_4}}{\partial y} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \longrightarrow \text{normale sortante } \boxed{\text{OK}}$$

$$I_{24} = \int_0^1 \int_0^{1-x} \underbrace{\vec{f}(\vec{\alpha_4}(x,y))}_{\begin{pmatrix} xy \\ y(1-(x+y)) \\ x(1-(x+y)) \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dy\,dx$$

Finalement :

$$I_2 = I_{24} = \int_0^1 \int_0^{1-x} xy + (x+y)(1-(x+y))\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} xy + (x+y) - (x+y)^2\, dy\, dx$$

$$= \int_0^1 x\left[\frac{y^2}{2}\right]_0^{1-x} + \left[\frac{(x+y)^2}{2}\right]_0^{1-x} - \left[\frac{(x+y)^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \frac{1}{2}\, x(1-x)^2 + \frac{1}{2} - \frac{x^2}{2} - \frac{1}{3} + \frac{x^3}{3}\, dx$$

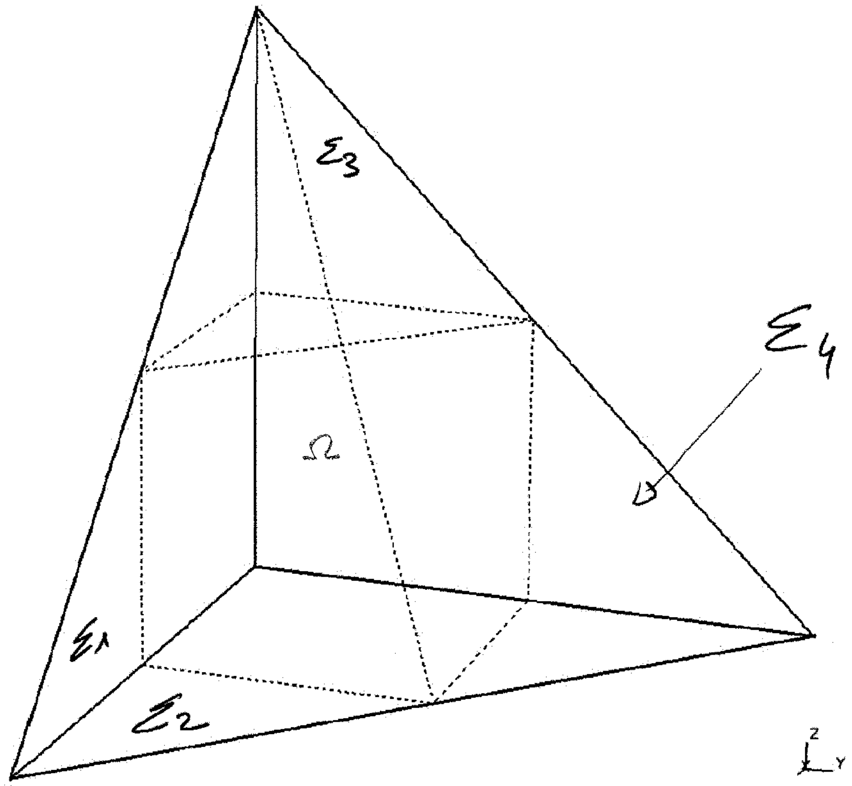
$$= \int_0^1 \frac{1}{2}\, x(1-2x+x^2)\, dx + \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{3} - \frac{1}{3} + \frac{1}{3}\cdot\frac{1}{4}$$

$$= \int_0^1 \frac{x}{2} - x^2 + \frac{x^3}{2}\, dx + \frac{1}{2} - \frac{1}{6} - \frac{1}{3} + \frac{1}{12}$$

$$= \frac{1}{2}\cdot\frac{1}{2} - \frac{1}{3} + \frac{1}{2}\cdot\frac{1}{4} + \frac{6-2-4+1}{12}$$

$$= \underbrace{\frac{1}{4} - \frac{1}{3}}_{-\frac{1}{12}} + \frac{1}{8} + \frac{1}{12} = \frac{1}{8}$$

$\longrightarrow$ OK le théorème de la divergence est vérifié.



Σ_3
Σ_4
Ω
Σ_1
Σ_2
Z
Y