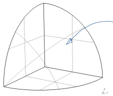
Ex 9)

(1)



Σ_1 ($\frac{1}{8}$ de la sphère unité)

(2) Ω est le huitième de la boule unité, donc son volume est :

$$V(\Omega) = \frac{1}{8} \cdot \frac{4\pi}{3} = \frac{\pi}{6}$$

(3) $\Sigma = \partial\Omega$, Σ_1 : partie de Σ correspondant à la surface courbe.

$$\Sigma_2 = \Sigma \setminus \Sigma_1$$

$$\vec{f}(x,y,z) = \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix}$$

Σ_1 :
$$\vec{r}_1(\varphi,\theta) = \begin{pmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{pmatrix}, \quad (\varphi,\theta) \in \left[0,\frac{\pi}{2}\right]^2$$

$$\Phi_1 = \iint_{\Sigma_1} \vec{f}\cdot d\vec{\Sigma}_1 = \int_0^{\pi/2}\int_0^{\pi/2} \vec{f}(\vec{r}_1(\varphi,\theta)) \cdot \left(\frac{\partial \vec{r}_1}{\partial\varphi} \wedge \frac{\partial \vec{r}_1}{\partial\theta}\right) d\varphi\, d\theta$$

$$\frac{\partial \vec{r}_1}{\partial\varphi} \wedge \frac{\partial \vec{r}_1}{\partial\theta} = \begin{pmatrix} \cos\varphi\cos\theta \\ \cos\varphi\sin\theta \\ -\sin\varphi \end{pmatrix} \wedge \begin{pmatrix} -\sin\varphi\sin\theta \\ \sin\varphi\cos\theta \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sin^2\varphi\cos\theta \\ \sin^2\varphi\sin\theta \\ \cos\varphi\sin\varphi \end{pmatrix}$$

$$\Phi_1 = \int_0^{\pi/2}\int_0^{\pi/2} \begin{pmatrix} \sin^3\varphi\cos^3\theta \\ \sin^3\varphi\sin^3\theta \\ \cos^3\varphi \end{pmatrix} \cdot \begin{pmatrix} \sin^2\varphi\cos\theta \\ \sin^2\varphi\sin\theta \\ \cos\varphi\sin\varphi \end{pmatrix} d\varphi\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \underbrace{\sin^5\varphi\cos^4\theta + \sin^5\varphi\sin^4\theta}_{\sin^5\varphi(\cos^4\theta + \sin^4\theta)} + \cos^4\varphi\sin\varphi\, d\varphi\, d\theta$$

$$= \underbrace{\left(\int_0^{\pi/2} \sin^5\varphi\, d\varphi\right)}_{I_2} \cdot \left(\underbrace{\int_0^{\pi/2} \cos^4\theta\, d\theta}_{I_3} + \underbrace{\int_0^{\pi/2} \sin^4\theta\, d\theta}_{I_4}\right)$$
$$+ \frac{\pi}{2} \cdot \underbrace{\int_0^{\pi/2} \cos^4\varphi\sin\varphi\, d\varphi}_{I_1}$$

- $I_1 = \left[-\frac{\cos^5\varphi}{5}\right]_0^{\pi/2} = \frac{1}{5}$

- $I_2 = \int_0^{\pi/2} \sin^5\varphi\, d\varphi = \int_0^{\pi/2} (\sin^2\varphi)^2 \sin\varphi\, d\varphi$

$$= \int_0^{\pi/2} (1-\cos^2\varphi)^2 \sin\varphi\, d\varphi$$

On pose $u = \cos\varphi$, $du = -\sin\varphi\, d\varphi$

$$I_2 = -\int_1^0 (1-u^2)^2\, du = \int_0^1 (1-u^2)^2\, du$$

$$= \int_0^1 1 - 2u^2 + u^4\, du$$

$$= 1 - 2\frac{1}{3} + \frac{1}{5} = \frac{15-10+3}{15} = \frac{8}{15}$$

- $I_3 = \int_0^{\pi/2} \cos^4\theta\, d\theta = \int_0^{\pi/2} (\cos^2\theta)^2\, d\theta$

$$= \int_0^{\pi/2} \frac{1}{4}(1+\cos 2\theta)^2\, d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} 1 + 2\cos 2\theta + \cos^2 2\theta\, d\theta$$

$$= \frac{1}{4}\left(\frac{\pi}{2} + \underbrace{\cancel{\left[\sin 2\theta\right]_0^{\pi/2}}}_{0} + \frac{1}{2}\int_0^{\pi/2} (1+\cos 4\theta)\, d\theta\right)$$

$$= \frac{\pi}{8} + \frac{1}{8}\cdot\left(\frac{\pi}{2} + \underbrace{\cancel{\left[\frac{\sin 4\theta}{4}\right]_0^{\pi/2}}}_{0}\right)$$

$$= \frac{\pi}{8} + \frac{\pi}{16} = \frac{3\pi}{16}$$

$$\circledast\ I_4 = \int_0^{\pi/2} \sin^4\theta \, d\theta = \int_0^{\pi/2} (\sin^2\theta)^2 d\theta$$

$$= \int_0^{\pi/2} \frac{1}{4}(1-\cos 2\theta)^2 d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} 1 - 2\cos 2\theta + \cos^2 2\theta \, d\theta$$

$$= \frac{1}{4}\left(\frac{\pi}{2} - \underbrace{\cancel{\left[\sin 2\theta\right]_0^{\pi/2}}}_{=0} + \frac{1}{2}\int_0^{\pi/2} 1 + \cos 4\theta \, d\theta\right)$$

$$= \frac{\pi}{8} + \frac{1}{8}\cdot\frac{\pi}{2} \quad + \frac{1}{2}\underbrace{\cancel{\left[\frac{\sin 4\theta}{4}\right]_0^{\pi/2}}}_{=0}$$

$$= \frac{3\pi}{16}.$$

Finalement : $\phi_1 = I_2(I_3 + I_4) + \frac{\pi}{2} I_1$

$$= \frac{8}{15}\underbrace{\left(\frac{3\pi}{16} + \frac{3\pi}{16}\right)}_{\frac{3\pi}{8}} + \frac{\pi}{2}\cdot\frac{1}{5}$$

$$= \frac{\pi}{5} + \frac{\pi}{10} = \frac{3\pi}{10}$$

(4) $\phi_2 = \iint_{\Sigma_2} \vec{g}\cdot d\vec{S} = \iint_{\Sigma} \vec{g}\cdot d\vec{S} - \underbrace{\iint_{\Sigma_1} \vec{g}\cdot d\vec{S}}_{\phi_1}$

Or : $\iint_\Sigma \vec{g}\cdot d\vec{S} = \iiint_\Omega \operatorname{div}\vec{g}\cdot dv$

$\operatorname{div}\vec{g} = 3x^2 + 3y^2 + 3z^2$

On paramètre Ω par :

$$\vec{\alpha}(r,\theta,\varphi) = \begin{pmatrix} r\sin\varphi\cos\theta \\ r\sin\varphi\sin\theta \\ r\cos\varphi \end{pmatrix}, \quad \begin{matrix} r\in[0,1] \\ \theta\in[0,\frac{\pi}{2}] \\ \varphi\in[0,\frac{\pi}{2}] \end{matrix}$$

$$|J(r,\theta,\varphi)| = \begin{vmatrix} \sin\varphi\cos\theta & -r\sin\varphi\sin\theta & r\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & r\sin\varphi\cos\theta & r\cos\varphi\sin\theta \\ \cos\varphi & 0 & -r\sin\varphi \end{vmatrix}$$

$$= \cos\varphi(-r^2\cos\varphi\sin\varphi) - r\sin\varphi(r\sin\varphi)$$

$$= -r^2\sin\varphi\,\underbrace{(\cos^2\varphi + \sin^2\varphi)}_{=1}$$

$$\iint_\Sigma \vec{g}\cdot d\vec{S} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \underbrace{3r^2}_{\operatorname{div}\vec{g}(\vec{\alpha}(r,\theta,\varphi))}(-r^2\sin\varphi)\, dr\, d\theta\, d\varphi$$

$$= \frac{3\pi}{2}\cdot\underbrace{\left[-\cos\varphi\right]_0^{\pi/2}}_{=1}\underbrace{\left[\frac{r^5}{5}\right]_0^1}_{1/5} = \frac{3\pi}{10}$$

Ainsi : $\phi_2 = \frac{3\pi}{10} - \frac{3\pi}{10} = 0$

(5) Σ_1 représente $\frac{1}{8}$ de la sphère unité, son aire est donc : $\frac{1}{8}\cdot 4\pi = \frac{\pi}{2}$

(6) Σ_2 représente 3 fois un quart de disque unité, son aire est donc :

$$3\times\frac{1}{4}\times\pi = \frac{3\pi}{4}.$$

(7) $\mathcal{C} = \partial\Sigma_1$

$$\int_{\mathcal{C}} \vec{g}\cdot d\vec{\ell} = \iint_{\Sigma_1} \vec{\operatorname{rot}}\,\vec{g}\cdot d\vec{S} = 0$$

$$\vec{\operatorname{rot}}\,\vec{g} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \wedge \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$